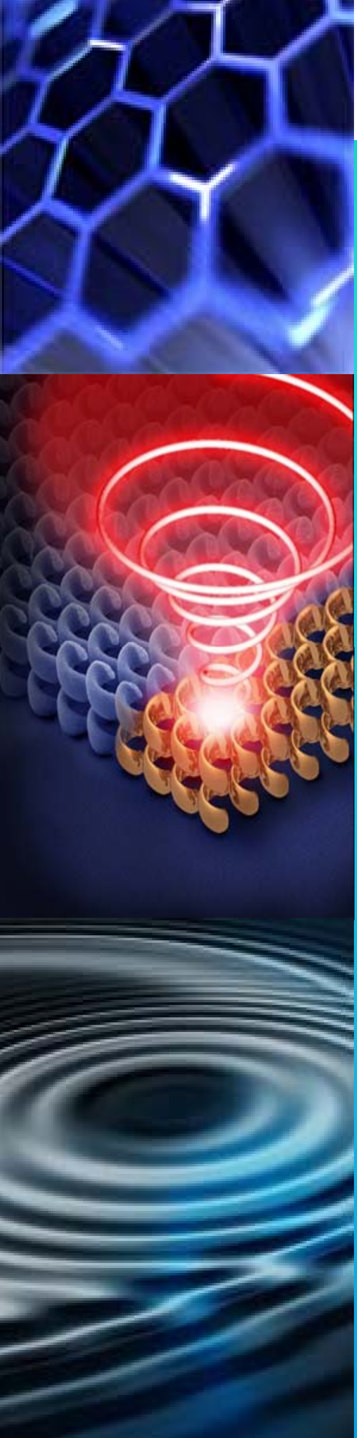

# Effective parameters of split-ring arrays, numerically determined by frequency-dependent homogenization

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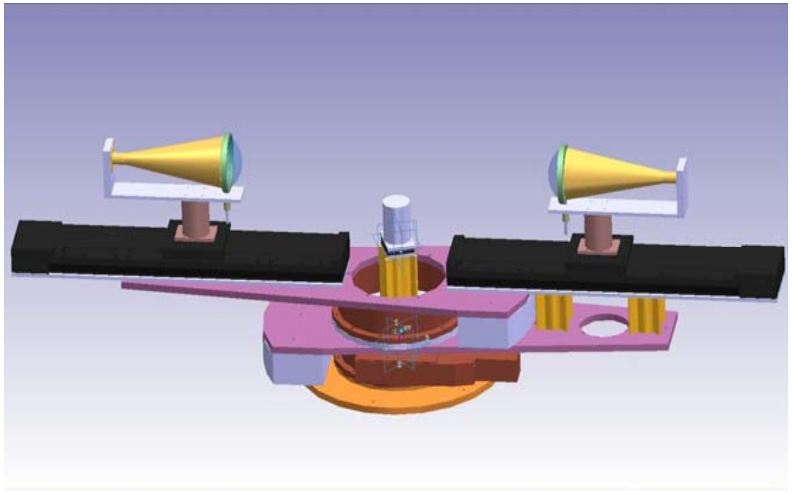
- An issue with the split-ring homogenization ?
- The weak formulation
- Model discretization
- Simulation results
- Conclusions

# Background : Free Space measurement of metamaterials

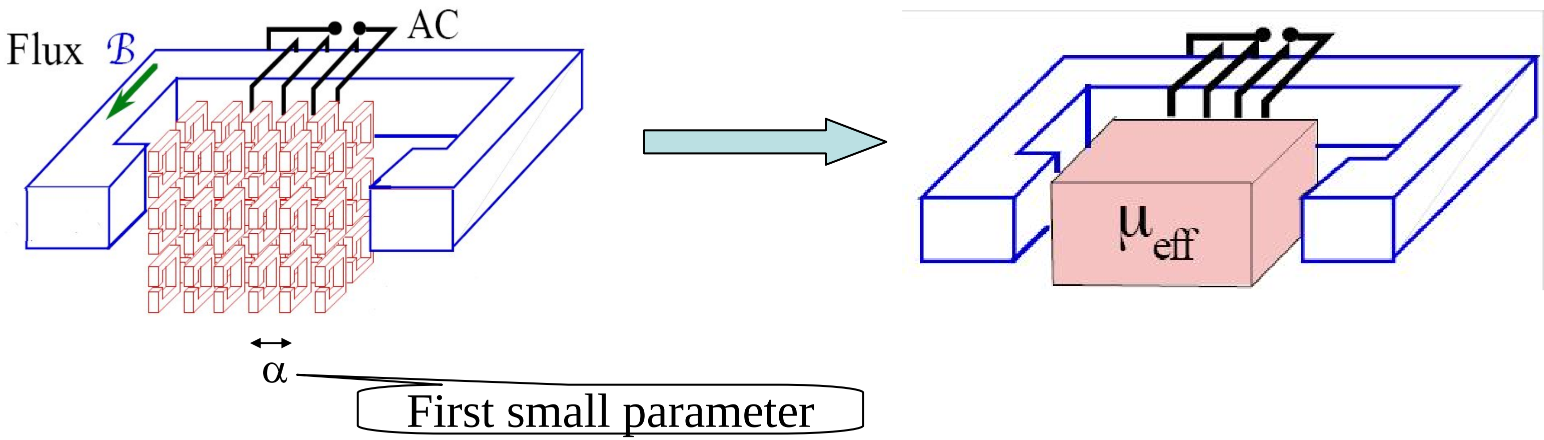


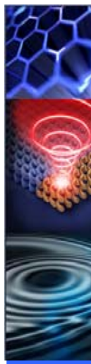
Operating in the 6-18 GHz frequency range

- Retrieve the S parameters.
- Compute $\varepsilon_{eff}$ and $\mu_{eff}$ with the Nicholson-Ross inversion.
- The incident wave on the structure is assumed to be planar.

- Metamatarials simulation would permit to define their dimensions for a resonance between 6 and 18 GHz.

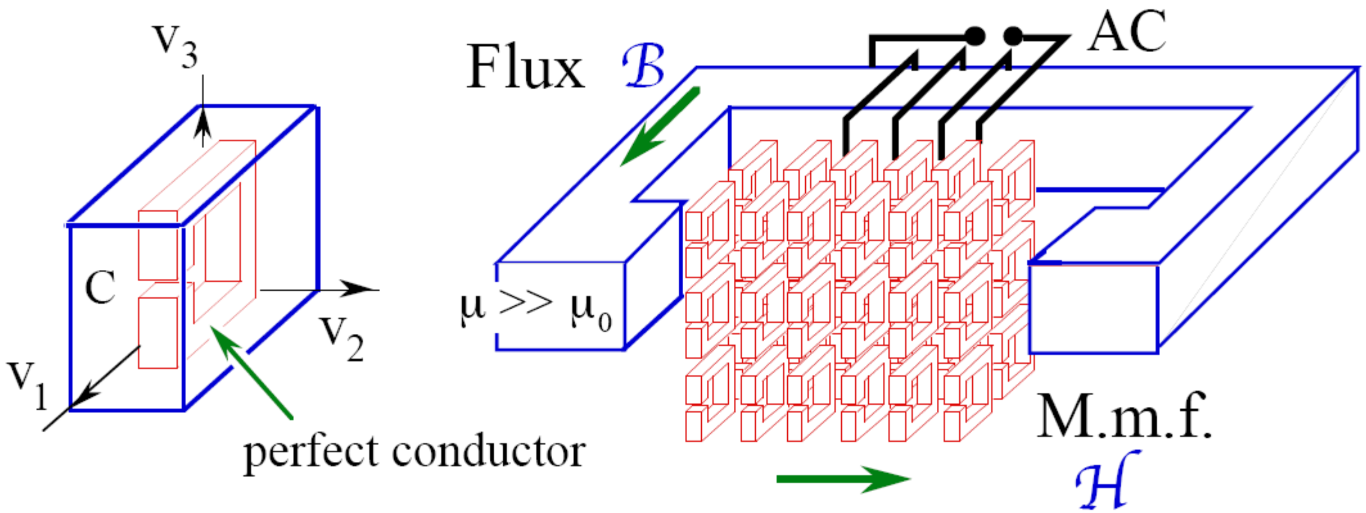
# The homogenization technique



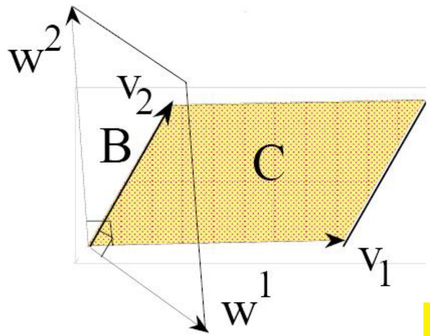
- Homogenization is the exploitation of translational symmetry of the periodic structure. Instead of solving the problem over the whole structure, we only study the symmetry cell.
- The incident wave « sees » the material as homogeneous when $\alpha << \lambda$.
- The computation time, needed memory and the mesh number of elements becomes reasonable.

# Introduction




- The split-ring array is immersed in a periodic field $Be^{i\omega t}$
- The Bravais lattice T is the set of translation vectors $\{\tau=\Sigma\tau_i V_i\}$
- The fields and the electromagnetic parameters are C-periodic $\mu(x+\tau)=\mu(x)$ (for each $\tau$ in T)

# Floquet-Block analysis




B, the "Brillouin zone", is the dual cell of C

$$v_i \cdot w^j = 2\pi\, \delta_i^j$$

$$\mathrm{vol}(B)\ \mathrm{vol}(C) = (2\pi)^3$$

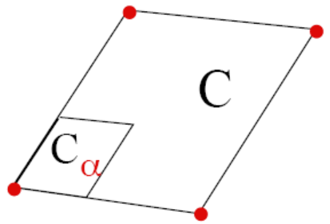
C-periodic

$$\hat{\varphi}_\kappa(x) = \mathrm{vol}(C) \sum_\tau e^{-i\kappa\cdot(\tau+x)}\varphi(x+\tau)$$

$$\varphi(x) = \frac{1}{(2\pi)^3}\int_B d\kappa\, e^{i\kappa\cdot x}\hat{\varphi}_\kappa(x)$$

- Bloch analysis consists in throwing $\varphi(x)$ into the equation to be solved, which results in a family of subproblems of the same kind, one for each wavevector $\kappa$.
- The homogenization simplification permits to solve all these subproblems in one stroke.

# Convergence and scaling




$$C_\alpha = \alpha C \qquad \text{(homothety)}$$

$$\lim_{\alpha \to 0} \langle \hat{\varphi}^\alpha_\kappa(x) \rangle_{C_\alpha} = \hat{\varphi}(\kappa)$$

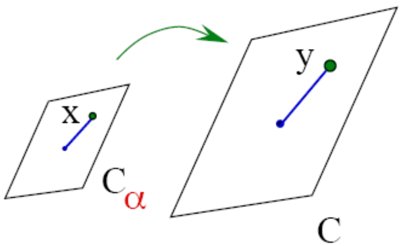
$$\Downarrow$$

$$\varphi^\alpha \rightharpoonup \varphi$$

$$h^\alpha_\kappa(y) \overset{\text{def.}}{=} \hat{h}^\alpha_\kappa(\alpha y)$$

$$\hat{h}^\alpha_\kappa(x) = h^\alpha_\kappa(x/\alpha)$$

$$\text{rot}\, \hat{h}^\alpha_\kappa(x) = \frac{1}{\alpha}\, \text{rot}\, h^\alpha_\kappa(y)$$




$$y = x/\alpha$$

To study the $\alpha = 0$ limit, one must be able to compare the $\hat{h}^\alpha_\kappa$ for *different* $\alpha$. "Pull back" to common domain C, by scaling, to let them all live on the *same* reference cell.

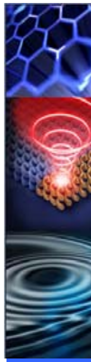
# Homogenizing the full Maxwell equations

$$-i\omega\vec{d} + rot\left(\vec{h}\right) = \vec{j}$$
$$-i\omega\vec{b} + rot\left(\vec{e}\right) = 0$$
$$\vec{d} = \varepsilon\vec{e}, \qquad \vec{b} = \mu\vec{h}$$

Block transformation $\Longrightarrow$

$$-i\omega\hat{d}_\kappa + (rot + i\kappa\times)\hat{h}_\kappa = \hat{j}_\kappa$$
$$-i\omega\hat{b}_\kappa + (rot + i\kappa\times)\hat{e}_\kappa = 0$$
$$\hat{d}_\kappa = \varepsilon\hat{e}_\kappa, \qquad \hat{b}_\kappa = \mu\hat{h}_\kappa$$

Scaling: $h^\alpha_\kappa(y) \overset{\text{def.}}{=} \widehat{h}^\alpha_\kappa(\alpha y)$, etc.



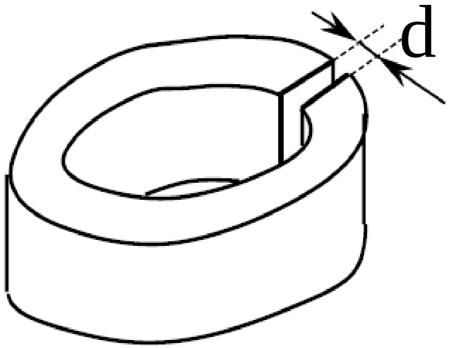
$$-i\omega\alpha d^\alpha_\kappa + (rot + i\alpha\kappa\times)h^\alpha_\kappa = \alpha j^\alpha_\kappa$$
$$d^\alpha_\kappa = \varepsilon\, e^\alpha_\kappa, \quad b^\alpha_\kappa = \mu\, h^\alpha_\kappa$$
$$i\omega\alpha b^\alpha_\kappa + (rot + i\alpha\kappa\times)e^\alpha_\kappa = 0$$

# Introducing a second small parameter

$$\mathrm{rot}\, h = 0 \qquad \langle h \rangle = H$$
$$b = \mu\, h \qquad B = \mu_{eff} H$$
$$\mathrm{div}\, b = 0 \qquad \langle b \rangle = B$$

b and h C-per.

$$-i\omega \hat{d}(\kappa) + i\kappa \times \hat{h}(\kappa) = \hat{j}(\kappa)$$
$$\hat{d}(\kappa) = \varepsilon_{eff}\, \hat{e}(\kappa)$$
$$i\omega \hat{b}(\kappa) + i\kappa \times \hat{e}(\kappa) = 0$$
$$\hat{b}(\kappa) = \mu_{eff}\, \hat{h}(\kappa)$$




$$d \prec\prec \alpha \prec\prec \lambda$$

- If we only assume that $\alpha << \lambda$ , the homogenization gives the static effective parameters . Therefore, no negative index could be obtained.

- Introducing a second small parameter, the slit's width d solves the previous problem.

# The weak formulation

$$\int_{U-T} i\omega\mu h h' + \int_{U-T} \frac{1}{i\varepsilon\omega}(rot(h)-j)\cdot rot(h') + \int_{\Sigma} \frac{d}{i\varepsilon\omega}(rot(h)\cdot n)\cdot(rot(h')\cdot n) = 0 \quad \forall h'\, test\ \ field$$

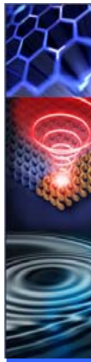
Scaling:

Gives rot(h)=0

$$\alpha^3 \int_{U-T} i\omega\mu h^{\alpha} h' + \alpha \int_{U-T} \frac{1}{i\varepsilon\omega}((rot + i\alpha\kappa\times)h^{\alpha} - \alpha\, j)\cdot(rot + i\alpha\kappa\times)\, h'^{\alpha}$$

$$+\, \alpha^3 \int_{\Sigma} \frac{d}{i\varepsilon\omega}((rot + i\alpha\kappa\times)h^{\alpha}\cdot n)\cdot((rot + i\alpha\kappa\times)\, h'^{\alpha}\cdot n) = 0 \quad \forall h'\, test\ \ field$$

Maintains the balance between the inductive and capacitive terms.

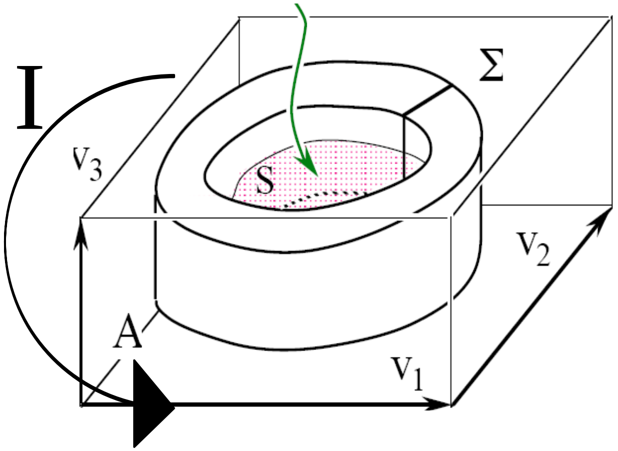


# The potential jump

*having* $\nabla \times h = 0$

$\Rightarrow \quad \exists \varphi \quad$ *locally as* $\quad h = \nabla \varphi$

We model the slit by a surface $\Sigma$ that bears a capacitive layer.
$\varphi$ is a multivalued magnetic potential.
$\varphi$ have a jump $[\varphi]=I$.




- Instead of meshing the small split, we introduce a cutting surface S through which the magnetic potential has a jump.
- Fewer tetrahedra are needed to mesh the unit cell.

# The weak formulation of the homogenized problem

$$\int_{U-T} \mu \nabla \varphi \nabla \varphi' dV \;+\; \int_{\partial T} \frac{1-i}{\sigma\omega\delta} \nabla_S \varphi \nabla_S \varphi' dS \;-\; \frac{1}{C\omega^2}[\varphi][\varphi'] \;=\; \int_{U-T} B \nabla \varphi' dV \qquad \forall \varphi' \in \Phi$$

$$C = \int_{\Sigma} \frac{\varepsilon}{d}$$

Joule losses

- Solve this linear system where the potential φ is defined on each node of the mesh.
- Several constrains for the nodes on the faces of the unit cell and for the nodes on the cutting surface

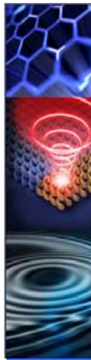
$$M\varphi = S$$

$$\varphi_{xa} - \varphi_{x0} = C_x$$

$$\varphi_{ya} - \varphi_{y0} = C_y$$

$$\varphi_{za} - \varphi_{z0} = C_z$$

$$\varphi_{s+} - \varphi_{s-} = I$$

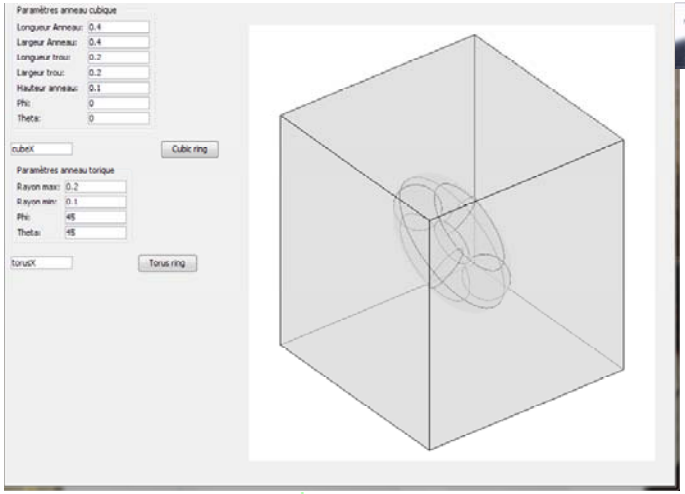


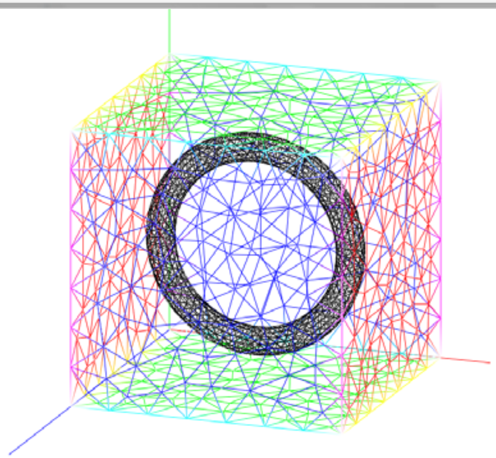
Energy of the equivalent cell

Energy of the actual cell

$$\frac{vol(U)}{\mu^*_{eff}} B^2 \;=\; \int_{U-T} \mu |\nabla \varphi|^2 \;+\; \int_{\partial T} \frac{1-i}{\sigma\omega\delta} |\nabla_S \varphi|^2 \;-\; \frac{1}{C\omega^2}[\varphi]^2$$

# Structure discretization: Meshing the model

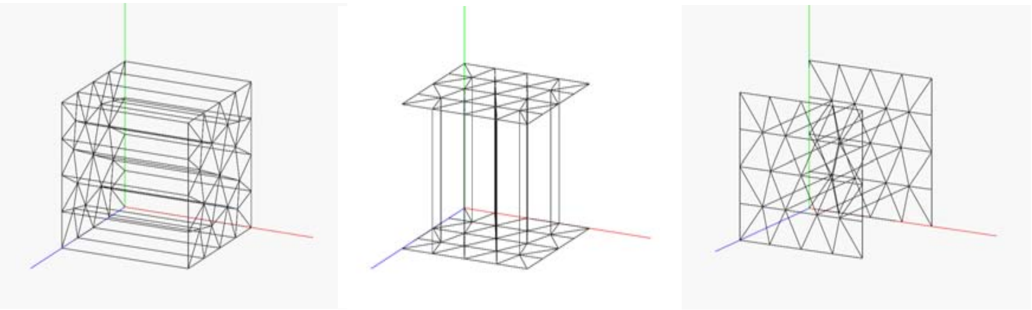


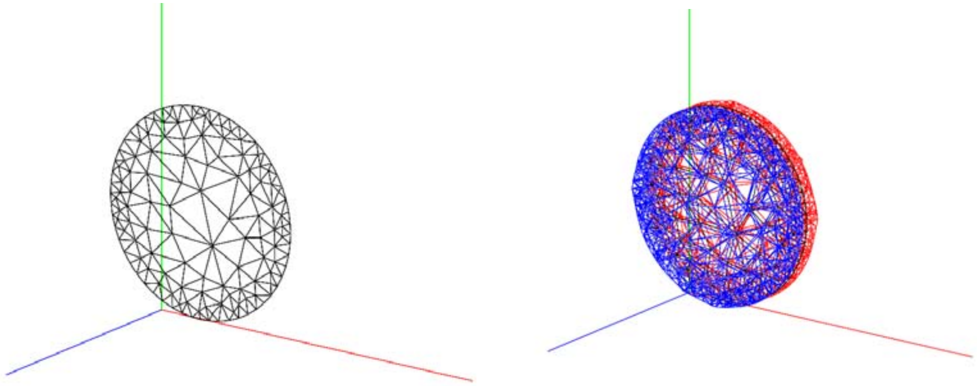
- The geometry and the periodic mesh are created under a COMSOL interface.

- The mesh is then transformed by a QT4 program to detect the unit cell periodicity and to double the nodes on S.

# Structure discretization: Node doubling



- Periodic structure: The opposite faces are meshed identically.
- Each node on a face of the unit cell is identified with a node on the opposite face.



- The nodes on the cutting surface are doubled an the connected elements to the cutting surface are transformed.
- The nodes are reordered.

# Nodes reordering

Potential on the other nodes

Conditions over the magnetic potential :

$$\begin{aligned} \varphi_{xa} - \varphi_{x0} &= C_x \\ \varphi_{ya} - \varphi_{y0} &= C_y \\ \varphi_{za} - \varphi_{z0} &= C_z \\ \varphi_{s+} - \varphi_{s-} &= I \end{aligned}$$

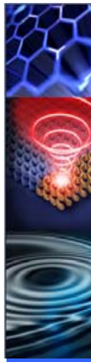
$$\begin{bmatrix} & \vdots & & \vdots & \\ & \vdots & & \vdots & \\ \cdots\ \cdots & M_{ii} & \cdots\ \cdots\ \cdots\ \cdots & M_{ij} & \cdots \\ & \vdots & & \vdots & \\ & \vdots & & \vdots & \\ & \vdots & & \vdots & \\ \cdots\ \cdots & M_{ji} & \cdots\ \cdots\ \cdots\ \cdots & M_{jj} & \cdots \\ & \vdots & & \vdots & \\ & \vdots & & \vdots & \end{bmatrix} \begin{bmatrix} \varphi_m \\ \varphi_{x0} \\ \varphi_{y0} \\ \varphi_{z0} \\ \varphi_{s+} \\ \varphi_{xa} \\ \varphi_{ya} \\ \varphi_{za} \\ \varphi_{s-} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ S_9 \end{bmatrix}$$

Potential of the unit cell faces x=0 and x=a nodes

where : $M_{ij} = {}^t\overline{M}_{ji}$

Potential of the cutting surface nodes

New linear system with the following unknowns : $\varphi_m\ \varphi_{x0}\ \varphi_{y0}\ \varphi_{z0}\ \varphi_{s+}\ C_x\ C_y\ C_z\ I$



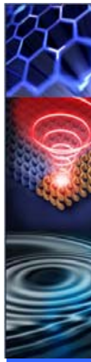
# Unknowns replacement : a simple case

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ {}^t\overline{M}_{12} & M_{22} & M_{23} \\ {}^t\overline{M}_{13} & {}^t\overline{M}_{23} & M_{33} \end{bmatrix} \begin{bmatrix} \varphi_m \\ \varphi_{x0} \\ \varphi_{xa} \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \qquad \varphi_{xa} - \varphi_{x0} = C_x$$

The matrix M being hermitian :

$$M\varphi = s$$

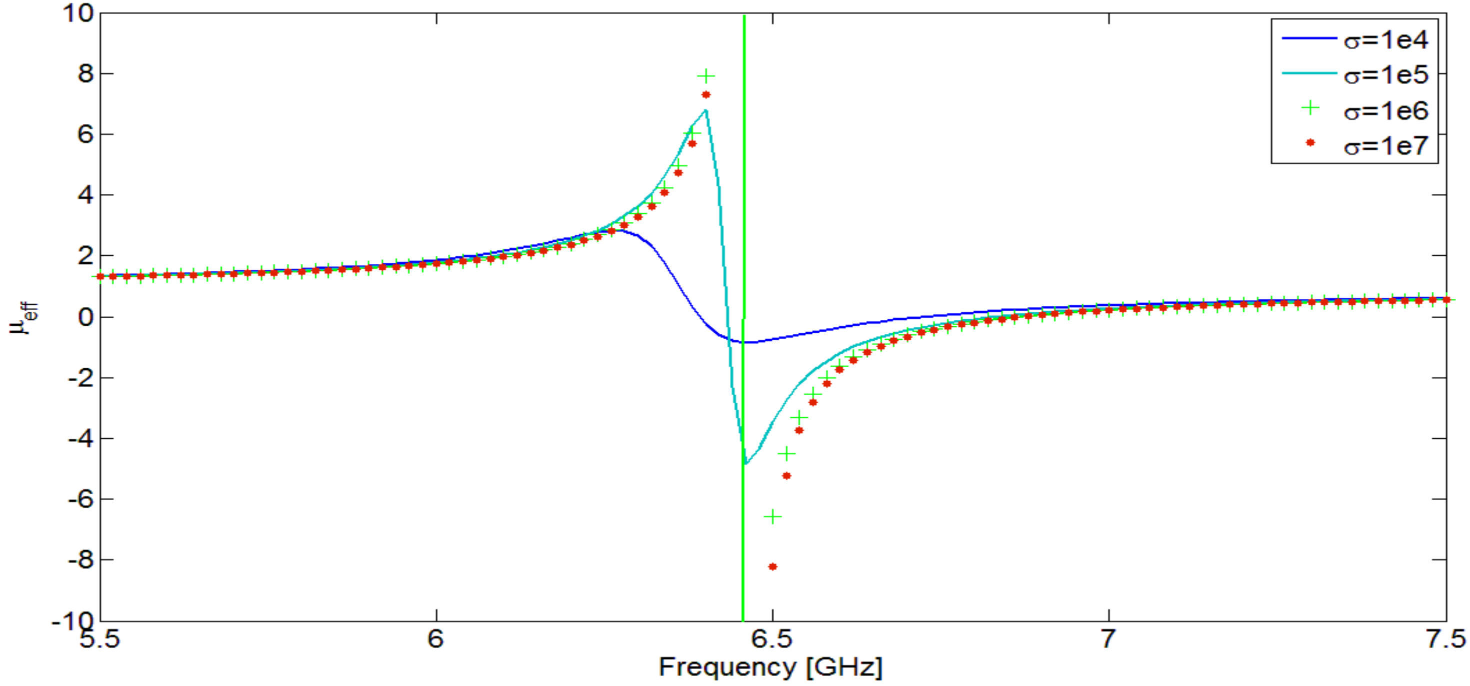
$${}^t\overline{\varphi}' M\varphi = {}^t\overline{\varphi}' s \qquad \forall \varphi'$$



$$\begin{bmatrix} {}^t\overline{\varphi}'_m & {}^t\overline{\varphi}'_{x0} & {}^t\overline{\varphi}'_{xa} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ {}^t\overline{M}_{12} & M_{22} & M_{23} \\ {}^t\overline{M}_{13} & {}^t\overline{M}_{23} & M_{33} \end{bmatrix} \begin{bmatrix} \varphi_m \\ \varphi_{x0} \\ \varphi_{xa} \end{bmatrix} = \begin{bmatrix} {}^t\overline{\varphi}'_m & {}^t\overline{\varphi}'_{x0} & {}^t\overline{\varphi}'_{xa} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

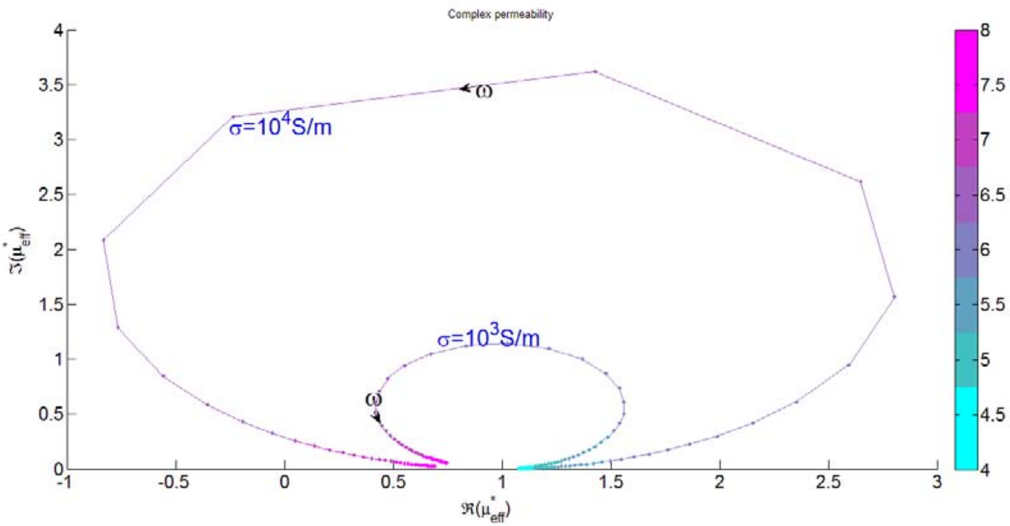
$$\begin{bmatrix} M_{11} & M_{12} + M_{13} & \sum_{cols} M_{13} \\ {}^t\overline{M}_{12} + {}^t\overline{M}_{13} & M_{22} + {}^t\overline{M}_{23} + M_{23} + M_{33} & \sum_{cols} M_{23} + M_{33} \\ \sum_{lines} {}^t\overline{M}_{13} & \sum_{lines} {}^t\overline{M}_{23} + M_{33} & \sum_{lines}\sum_{cols} M_{33} \end{bmatrix} \begin{bmatrix} \varphi_m \\ \varphi_{x0} \\ C_x \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 + s_3 \\ \sum_{lines} s_3 \end{bmatrix}$$

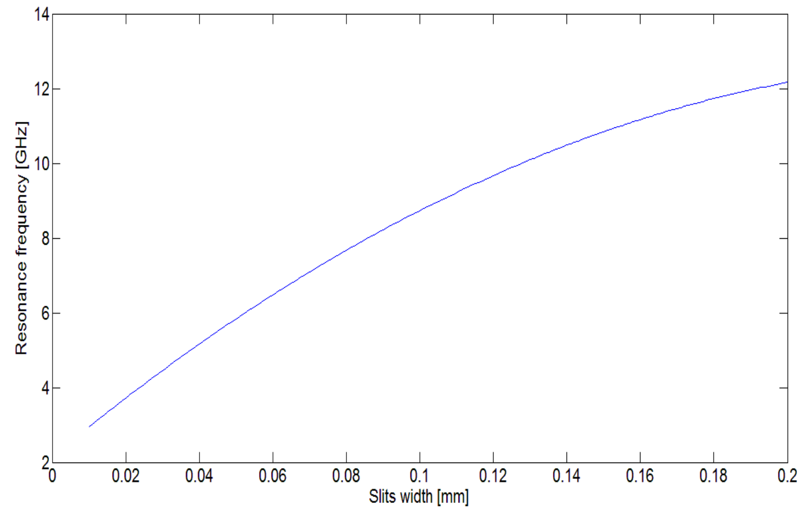
# Results : Effective permeability




- Copper cubic ring resonance at 6.25 GHz. Unit cell is $1cm^3$ and the slit's width is 0.1mm.
- The permeability and the current tends to infinity if the ring's is very conductive.

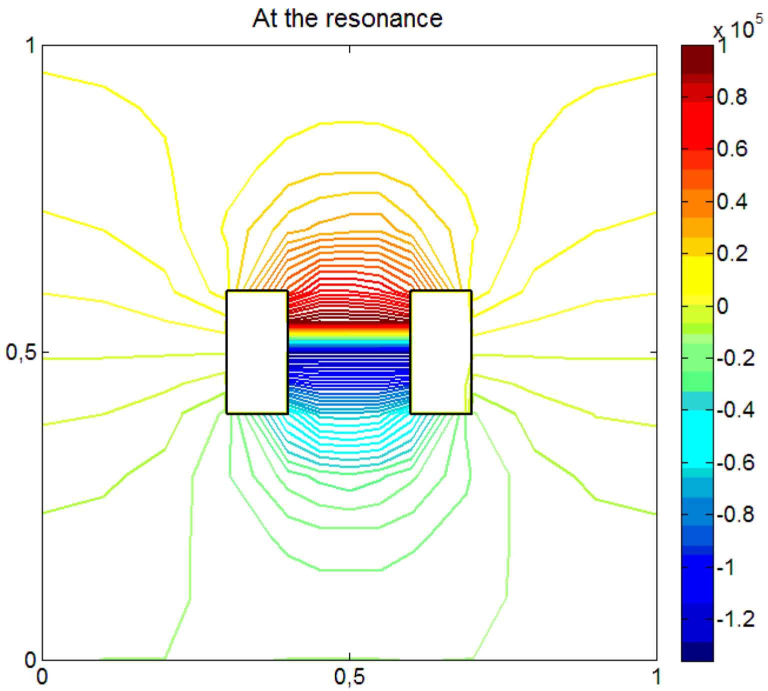
# Conductivity and slit's width incidence

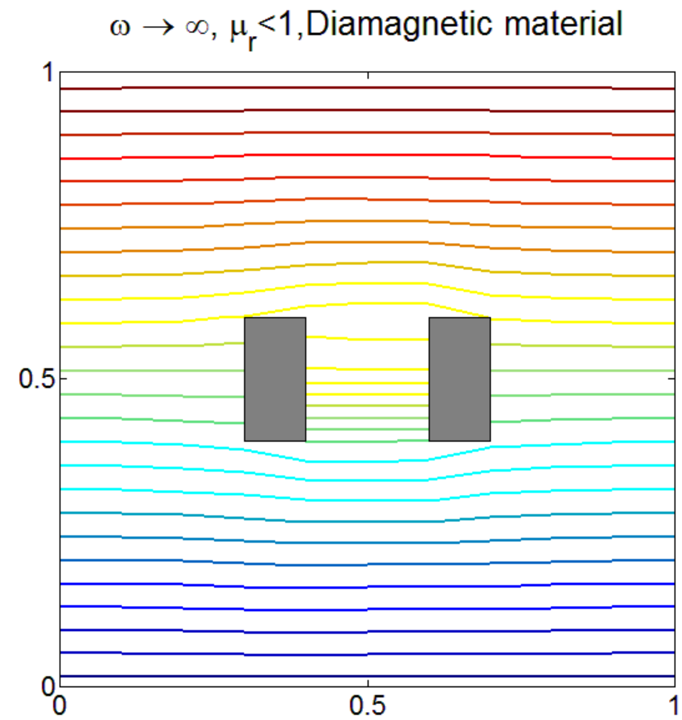


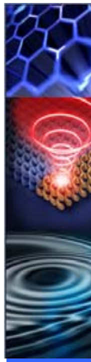




- The resonance frequency slightly decreases while the conductivity does.
- We assume a skin depth δ much smaller than the slit's width d.

# Isopotentials



- The magnetic field flows through the ring.
- The magnetic potential has a jump when passing through the cutting surface S.

# Conclusions

- The effective permeability, with a negative real part, is obtained with a minimal computational cost.
- The skin effect is taken into account while the number of the mesh elements is still reasonable.
- The automated creation of geometries and the short time simulation open the way to geometry optimization.

- A complex implementation with several mesh transformations.
- The ring is closed and the slit disappears. Therefore this model is not adapted to the simulation of double split-ring resonators.