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# Work of a time-dependent force[1]

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**Abstract**

Certain subtleties concerning the work done by a time-dependent force field are discussed. In particular, it is explained why such a field cannot be conservative even if it is irrotational and its region of action has the proper topological properties.

## 1. Introduction

In a previous article [1] a common misconception regarding the electromotive force (emf) of electrodynamics was discussed. Specifically, it was explained why it is incorrect to *define* the emf as work (per unit charge), in general. In simple terms, the emf is always determined for a given instant of time, whereas in determining the work of a force field on a particle (here, an electric charge) moving along a space curve, time is allowed to flow during the motion. Of course, there *are* exceptional situations where the emf of a circuit does indeed coincide in value with work per unit charge for a complete tour around the circuit [1].

From the point of view of classical mechanics the case of time-dependent forces and their work constitutes an interesting problem. In the present article we highlight certain aspects of this problem, focusing on subtleties that arise when one goes beyond the comfortable case of static force fields. Of course, the subject of time-dependent forces and associated potentials is discussed in many standard textbooks of mechanics (see, e.g., [2-5]). Our aim here is to extend the discussion in these sources by adding a few comments that may help the student to further clarify the situation.

In Section 2 we define the work done by a time-dependent force field on a test particle and point out certain subtle points of this definition.

In Sec. 3 we discuss the relation between irrotational and conservative force fields. We explain why time-dependent fields cannot be conservative and do not lead to conservation of total mechanical energy.

## 2. Work along a space curve

Consider a test particle of mass $m$ moving in a region of space permeated by a force field $\vec{F}$. The particle is assumed to move along a space curve $L$ extending from point $A$ to point $B$ (Fig. 1). We call $\vec{r}$ the position vector of $m$ on $L$ at time $t$, relative to the origin $O$ of some inertial reference frame, and we denote by $d\vec{r}$ the elementary displacement of $m$ along $L$ in an infinitesimal time interval $dt$.

[1] This article is an addendum to the published article [1].

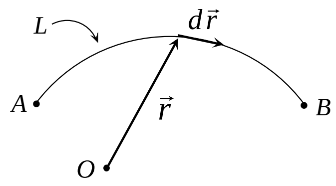


Figure 1

The *work* done by the field $\vec{F}$ on *m* from *A* to *B* is

$$W = \int_L \vec{F} \cdot d\vec{r} \qquad (1)$$

To compute the line integral in (1) one needs to have a mathematical description of the curve *L*. Of course, a parametric representation of *L* is possible by using any convenient parameter whose values correspond to the various points $\vec{r}$ of *L*. However, a mere geometrical description of *L* may not be sufficient in order to specify the work *W*, since it may be important to take into account the *time* at which the particle *m* passes through any given point of the curve. Thus, the most faithful parameterization of *L* in this regard is provided by the *equation of motion* of *m*, connecting the position $\vec{r}$ of the particle with the time *t* at which the particle passes from that position.

Let us assume the following mathematical description of the motion of *m* along the trajectory *L*:

$$\vec{r} = \vec{\phi}(t) \;\; ; \;\; t_0 \le t \le t_1 \quad \text{with} \quad \vec{\phi}(t_0) = \vec{r}_A \;,\; \vec{\phi}(t_1) = \vec{r}_B \qquad (2)$$

Then, $d\vec{r} = d\vec{\phi}(t) = \vec{\phi}'(t)\, dt$ . The complexity of the integration (1) now depends on the nature of the force field $\vec{F}$ ; specifically, the dependence or not of this field on time.

For a *static* force field $\vec{F}(\vec{r})$, we have:

$$W = \int_{t_0}^{t_1} \vec{F}\left(\vec{\phi}\left(t\right)\right) \cdot \vec{\phi}'(t)\, dt \qquad (3)$$

This quantity is *independent of the parameterization* of the curve *L*, i.e., independent of the specific functional dependence of $\vec{r}$ on *t* as expressed by (2). Indeed, the substitution $\vec{\phi}(t) = \vec{r}$ transforms the integral (3) into

$$W = \int_A^B \vec{F}(\vec{r}) \cdot d\vec{r} \qquad (4)$$

Evidently, the integral on the right depends only on the geometry of the space curve *L*, not on the specific parameterization of this curve. In conclusion,

> *in a static force field, work is a well-defined quantity depending on the path followed by the particle in the field.*

Things become a lot more complicated in the case of a *time-dependent* force field $\vec{F}(\vec{r}, t)$. The work on the particle *m* along the curve *L* is written

$$W=\int_L \vec{F}\cdot d\vec{r}=\int_A^B \vec{F}(\vec{r},t)\cdot d\vec{r} \qquad (5)$$

It should be noted carefully that, inside the integral, the variables $\vec{r}$ and $t$ are not independent of each other since the former is a function of the latter through the parameterization (2) of $L$, i.e., in accordance with the specific equation of motion of $m$ along $L$. Relation (5) is written

$$W=\int_{t_0}^{t_1} \vec{F}\left(\vec{\phi}(t),t\right)\cdot\vec{\phi}'(t)\,dt \qquad (6)$$

This time the substitution $\vec{\phi}(t)=\vec{r}$ will not eliminate $t$ in favor of $\vec{r}$. Thus, the work $W$ is no longer independent of the parameterization of the curve $L$ by the equation of motion of $m$. The sole geometry of $L$ is not sufficient in order to determine $W$!

To understand this better, consider the elementary work $dW=\vec{F}\cdot d\vec{r}$. In the case of a static force field, this is written $dW=\vec{F}(\vec{r})\cdot d\vec{r}$. For a given equation of motion of the form (2), $dW$ depends only *implicitly* on $t$ through the relation $\vec{r}=\vec{\phi}(t)$. Thus, for a given elementary displacement of the particle along $L$, $dW$ depends solely on the position $\vec{r}$ of $m$ on the curve, not on the time at which the particle passes by that position. As $t$ varies from $t_0$ to $t_1$, the position vector $\vec{r}$ traces out all curve points from $A$ to $B$. Eventually, the total work $W$, given by (4), has a well-defined value independent of the parameterization of $L$. This work depends only on the geometry of the trajectory $L$ connecting $A$ and $B$.

On the other hand, in the case of a time-dependent force field the elementary work is of the form $dW=\vec{F}(\vec{r},t)\cdot d\vec{r}$. Here, $dW$ depends *explicitly* on $t$. Thus, for a given elementary displacement along $L$, $dW$ depends not only on the position of the particle on $L$ but also on the time the particle passes from that position. This, in turn, depends on the equation of motion $\vec{r}=\vec{\phi}(t)$, i.e., on the specific parameterization of $L$. Therefore the total work (5) is not a uniquely defined quantity but depends on the equation of motion along $L$.

## 3. Conservative and irrotational fields

Let $\vec{F}(\vec{r})$ be a static force field. Generally speaking, this field is *conservative* if the work it does on a test particle $m$ is path-independent, or equivalently, if

$$\oint_C \vec{F}(\vec{r})\cdot d\vec{r}=0 \qquad (7)$$

for any closed path $C$ within the field.

Let $S$ be an open surface bounded by a given closed curve $C$ in the field (Fig. 2). By Stokes' theorem and by Eq. (7),

$$\oint_C \vec{F}(\vec{r})\cdot d\vec{r}=\int_S (\vec{\nabla}\times\vec{F})\cdot\overrightarrow{da}=0 \qquad (8)$$

In order for this to be true for every *S* bounded by *C*, the field $\vec{F}(\vec{r})$ must be *irrotational*:

$$\vec{\nabla}\times\vec{F}=0 \qquad (9)$$

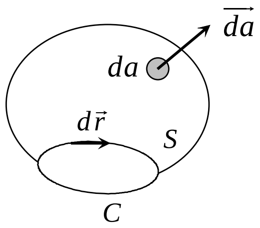


Figure 2

Conversely, an irrotational force field $\vec{F}(\vec{r})$ will also be conservative in a region of space that is *simply connected* [6,7]. Indeed, given any closed curve *C* in such a region, it is always possible to find an open surface *S* having *C* as its boundary. Then, if (9) is valid, the force is conservative in view of (8).

Given a conservative force field $\vec{F}(\vec{r})$, there exists a function $U(\vec{r})$ (*potential energy* of the particle *m*) such that

$$\vec{F}=-\vec{\nabla}U \qquad (10)$$

The work *W* from point *A* to point *B* in the field is then equal to

$$W=\int_A^B \vec{F}(\vec{r})\cdot d\vec{r}=U(\vec{r}_A)-U(\vec{r}_B) \qquad (11)$$

As is well known (and as will be shown analytically below) the *total mechanical energy* of *m* is constant during the particle’s motion inside the force field. This energy is the sum $E=T+U$ of the kinetic energy $T=mv^2/2$ (where *v* is the speed of the particle) and the potential energy *U*.

Consider now a time-dependent force field $\vec{F}(\vec{r},t)$ in a simply connected region *Ω* of space. This field is assumed to be irrotational for all values of *t*:

$$\vec{\nabla}\times\vec{F}(\vec{r},t)=0 \qquad (12)$$

Can we conclude that the field $\vec{F}$ is conservative?

It is tempting but *incorrect* (!) to argue as follows: Let *C* be an arbitrary closed curve in *Ω*. Since *Ω* is simply connected, there is always an open surface *S* bounded by *C*. By Stokes’ theorem,

$$\oint_C \vec{F}(\vec{r},t)\cdot d\vec{r}=\int_S(\vec{\nabla}\times\vec{F})\cdot\overrightarrow{da}=0 \qquad (13)$$

for all values of *t*. This *appears* to imply that $\vec{F}$ is conservative. This is not so, however, for the following reason: For any fixed value of *t*, the integral

$$I(t) = \oint_C \vec{F}(\vec{r},t) \cdot d\vec{r}$$

does *not* represent work. Indeed, $I(t)$ expresses the integration of a function of two independent variables, $\vec{r}$ and $t$, over one of these variables (namely, $\vec{r}$), the other variable ($t$) playing the role of a "parameter" of integration which remains fixed. Thus, $I(t)$ is evaluated *for a given instant of time t* and all values of $\vec{F}$, at the various points of $C$, must be recorded simultaneously at $t$.

On the other hand, in the integral representation of work,

$$W = \oint_C \vec{F}(\vec{r},t) \cdot d\vec{r} \ ,$$

time is assumed to flow as the test particle $m$ travels along the closed curve $C$. In this case, $\vec{r}$ and $t$ are no longer independent of each other but are connected through the equation of motion of $m$ on $C$, which equation mathematically endows $C$ with a certain parameterization. This complication never arises in the case of static fields, as we saw previously. We may thus conclude that

> *a force field that is both static and irrotational in a simply connected region of space is conservative; a time-dependent force field cannot be conservative even if it is irrotational and its region of action is simply connected.*

Finally, let us explain why a time-dependent force field does not lead to conservation of total mechanical energy. Consider again an irrotational force field $\vec{F}(\vec{r},t)$ [as defined according to (12)] in a simply connected region $\Omega$. Then there exists a time-dependent potential energy $U(\vec{r},t)$ of $m$, such that, for any value of $t$,

$$\vec{F}(\vec{r},t) = -\vec{\nabla} U(\vec{r},t) \qquad (14)$$

This time we will assume that $\vec{F}(\vec{r},t)$ is the *total* force on $m$. By Newton's 2nd law, then,

$$m\frac{d\vec{v}}{dt} = \vec{F} \quad (\text{where } \vec{v} = d\vec{r}/dt) \ \Rightarrow \ m\frac{d\vec{v}}{dt} + \vec{\nabla} U = 0 \ .$$

Taking the dot product with $\vec{v}$, we have:

$$m\vec{v} \cdot \frac{d\vec{v}}{dt} + \vec{v} \cdot \vec{\nabla} U = 0 \ .$$

Now,

$$\vec{v} \cdot \frac{d\vec{v}}{dt} = \frac{1}{2}\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{1}{2}\frac{d}{dt}(v^2) \quad (v = |\vec{v}|)$$

and

$$\vec{v} \cdot \vec{\nabla} U = \frac{\vec{\nabla} U \cdot d\vec{r}}{dt} = \frac{dU - \frac{\partial U}{\partial t}dt}{dt} = \frac{dU}{dt} - \frac{\partial U}{\partial t}$$

where we have used the fact that $dU(\vec{r},t)=\vec{\nabla}U\cdot d\vec{r}+\frac{\partial U}{\partial t}dt$ . Hence, finally,

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right)+\frac{dU}{dt}-\frac{\partial U}{\partial t}=0 \;\Rightarrow$$

$$\frac{d}{dt}(T+U)=\frac{\partial U}{\partial t} \qquad (15)$$

where $T=mv^2/2$. As seen from (15), the total mechanical energy ($T+U$) of $m$ is not conserved unless $\partial U/\partial t=0$, i.e., unless the force field is static.

Note that, for a time-dependent irrotational force field [defined according to (12)] the quantity

$$\int_A^B \vec{F}(\vec{r},t)\cdot d\vec{r} = U(\vec{r}_A,t)-U(\vec{r}_B,t)\,,$$

defined for any *fixed t*, does *not* represent the work done by this field on a particle $m$ from $A$ to $B$ [comp. (11) for the case of a static force field]. That is,

> *the work of a time-dependent irrotational force field cannot be expressed as the (negative) difference of the values of the corresponding time-dependent potential energy at the end points of the trajectory of a particle.*

## 4. Summary

Let us summarize our main conclusions:

1. In a static force field, the work done on a test particle is a well-defined quantity that depends on the geometrical characteristics of the particle's trajectory in the field.

2. In a time-dependent force field, the geometry of the trajectory is not sufficient in order to determine work: one must also know the precise equation of motion of the particle along this trajectory, connecting the position of the particle with time. Thus, work is not a uniquely defined quantity in this case.

3. A static force field that is irrotational in a simply connected region of space is conservative.

4. A time-dependent force field cannot be conservative even if it is irrotational and its region of action has the proper topology.

5. The work of a time-dependent irrotational force field cannot be expressed as the difference of the values of the time-dependent potential energy at the end points of the trajectory of a particle.

6. Time-dependent force fields are incompatible with conservation of total mechanical energy.

## References


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