A convergence proof for dynamic homogenization

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Abstract— We justify, by an asymptotic analysis, an homogenization method which applies to the Maxwell system of equations in the harmonic case at angular frequency ω. The method yields effective coefficients by solving the Maxwell system on a fundamental domain of the crystal-like metamaterial of interest, with appropriate periodic boundary conditions. The case of all-dielectric materials, with strong contrast, is especially addressed.

1. INTRODUCTION

'Dynamic homogenization' [1] should be understood, in this paper, as a procedure that delivers *frequency-dependent* effective coefficients $\epsilon_{\text{eff}}(\omega)$ and $\mu_{\text{eff}}(\omega)$ from a complete description of the local values of ϵ and μ inside the periodicity cell of a metamaterial. 'Static' homogenization, in contrast, is the kind of (non-trivial) spatial averaging of ϵ and μ described in classic work such as [2], which works fine in static situations, but cannot take into account the internal resonance phenomena that make metamaterials what they are.

In both cases, one first solves a so-called 'cell problem', on a periodicity cell of the material, with periodic boundary conditions, from which would-be effective coefficients are computed. One then replaces the macroscopic body, made of this material, by a homogeneous medium of the same shape, endowed with these effective coefficients, other elements of the situation (field generator(s), other homogeneous bodies present, air around, etc.), being unchanged.

Such a procedure needs justification, in the form of an appropriate asymptotic result: "When the size of the periodicity cell tends to 0, the solution of the Maxwell equations (over the whole setup) converges, in some sense, towards ...", etc. This is relatively easy (and now well known) in the static case, but much more of a challenge in dynamic homogenization, when all fields are time-harmonic at some definite angular frequency ω.

We propose here a way to state and prove such things in the dynamic case for a large family of metamaterial designs. (The case of bispherical dielectric inclusions [3, 4, 5], with strong contrast over the dielectric background, will serve as concrete example.) The main idea is that internal resonance phenomena should somehow be 'kept invariant' in the asymptotic analysis, in spite of the fact that letting the periodicity cell shrink to a point tends to move such resonances away from ω. Relations between Floquet–Bloch analysis (on periodic media) and Fourier analysis (over homogeneous ones) play a key role in the proofs [6].

2. DETAILS

Consider a lattice generated by three independent vectors $\partial_1, \partial_2, \partial_3$. Call C (the periodicity cell) the parallelepiped built on them, and suppose 3D space filled by a material for which $\mu(x+v) = \mu(x)$ and $\epsilon(x+v) = \epsilon(x)$ for all points x and all vectors v belonging to the lattice (a so-called "C-periodic" material). A source current, $j^s(x)$ at point x, is given.

Suppose j^s time-independent. Magnetostatics consists in solving for h and b such that $\operatorname{div} b = 0$, $b = \mu h$, $\operatorname{rot} h = j^s$. Call this "problem P_1". It can be embedded, conceptually, in a family of similar virtual problems P_α, for which the cell is C_α generated by the vectors $\alpha\partial_1, \alpha\partial_2, \alpha\partial_3$, with j^s unchanged. Using Bloch decomposition (relative to C_α), one can prove [6] that the solution (b_α, h_α) weakly converges, when the non-dimensional parameter α tends to 0, towards the solution (b_0, h_0) of the following "problem P_0":

$$\operatorname{div} b = 0, \quad b = \mu_{\text{eff}}\, h, \quad \operatorname{rot} h = j^s, \tag{1}$$

where μ_{eff} is the 3×3 matrix defined, for any nonzero vector$H$, by

$$\mu_{\text{eff}} H \cdot H = [\inf_\varphi \{ \int_C \mu |H + \operatorname{grad} \varphi|^2 \}]/\text{volume}(C), \tag{2}$$

where the infimum is taken with respect to all (smooth enough) C-periodic magnetic potentials φ living on C.

In practice, the material with C-periodic μ has finite extent, but this convergence result is enough to justify *homogenization*, i.e., the replacement of μ by μ_{eff} in the bulk metamaterial. In the dynamic case, where the excitation current at time t and point x is $\mathrm{Re}[j^s(x)\exp(i\omega t)]$, the analogous procedure consists in solving the homogenized Maxwell equations (all fields b, h, etc., now complex-valued):

$$-i\omega d + \mathrm{rot}\, h = j^s, \quad d = \epsilon_{\text{eff}}\, e, \;\; b = \mu_{\text{eff}}\, h, \quad i\omega b + \mathrm{rot}\, e = 0 \tag{3}$$

where again ϵ_{eff} and μ_{eff} come from solving a preliminary cell problem, as follows. (We ignore the possibility of chiral behavior for the moment, see below.)

Let us denote by $U \times x$, where U is a given (complex) 3D vector, the vector *field* whose value at point x is the cross product $U \times \overrightarrow{cx}$, where c is the center of the cell C and $\overrightarrow{cx}$ the vector from c to x. (Remark that $\mathrm{rot}(U \times x) = 2U$.) Now, given 3D vectors B and D, we solve on C for fields b, h, d, e such that $e + \frac{i\omega}{2}\, B \times x$ and $h - \frac{i\omega}{2}\, D \times x$ be C-periodic and

$$-i\omega\epsilon\, e + \mathrm{rot}\, h = 0, \quad i\omega\mu\, h + \mathrm{rot}\, e = 0 \;\text{ in } C. \tag{4}$$

(The C-periodicity conditions concern only the tangential components of the fields on the cell's boundary, and are easily dealt with, numerically, when using edge-based finite elements, cf. [7].) This done, we compute the following Lagrangian, a quadratic function of B and D:

$$\mathcal{L}(B, D) = (\textstyle\int_C [\mu\, h \cdot \overline{h} - \overline{\epsilon}\, \overline{e} \cdot e])/\mathrm{volume}(C), \tag{5}$$

hence two (a priori, complex) vectors H and E, linearly depending on B and D, such that $\mathcal{L}(B, D) = B \cdot \overline{H} - \overline{D} \cdot E$. The relation thus found between the pairs (B, D) and (H, E) is the desired homogenized law. Its general form is $B = \mu_{\text{eff}}\, H + \zeta_{\text{eff}}\, E$ and $D = \xi_{\text{eff}}\, H + \epsilon_{\text{eff}}\, E$, where μ_{eff}, ζ_{eff}, etc., are 3×3 complex matrices which do depend on ω. Note the possibility of anisotropy and of chirality (nonzero ζ_{eff} and ξ_{eff}).

We justify the procedure by a convergence result in the case (important in metamaterial studies, cf. [3, 4, 5]) when C contains a region C_1 in which $\epsilon(x) = \epsilon_1$, a large value with respect to ϵ_0, and $\epsilon(x) = \epsilon_0$ for x in $C - C_1$. Call P_1, again, the problem, set in the whole space,

$$-i\omega\epsilon\, e + \mathrm{rot}\, h = j^s, \quad i\omega\mu_0\, h + \mathrm{rot}\, e = 0. \tag{6}$$

The trick consists in embedding P_1 in a family P_α of similar problems, with $C^\alpha = \alpha C$ as periodicity cell, $\epsilon = \epsilon_0$ in $C_\alpha - \alpha\, C_1$, and $\epsilon = \epsilon_1/\alpha^2$ in αC_1. We also embed the cell problem (4)—call it CP_1—in a family CP_α set on the cell C (thanks to a rescaling move), with the same modified values of ϵ inside. The same Fourier–Bloch analysis as in the static case leads to a cell problem which is precisely the limit of CP_α, hence the desired asymptotic result. The 'contrast enhancement', so to speak, operated by dividing ϵ_1 by α^2, is the way by which one can keep the resonance values close to ω, as $\alpha \to 0$, in this category of problems with dielectric inclusions at strong contrast over the background.

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