Effective, Frequency-Dependent, Permeability of "Split-Ring" Metamaterials via Homogenization

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Abstract–**Homogenization, which reduces the cost of numerical simulations in materials with repetitive structure, is a promising approach to the design of metamaterials. We justify the procedure by a convergence result and apply it to the case of an array of split rings, where a *negative* effective permeability can be expected, on physical grounds, near some resonant frequency. A simplified model confirms this.**

Index Terms–**Homogenization, metamaterials, symmetry.**

I. A Case Study In Homogenization

When a regular, crystal-like, array of small "split rings" (Fig. 1, left) is immersed in an AC magnetic field, the "metamaterial" thus obtained can behave in surprising ways [1]. For instance, the spatial average B of b (at a large enough scale with respect to the size of the "cell" C of Fig. 1), and the spatial average P of the reactive power inside the material relate by $P = i\omega\ (\nu_{eff} B)^* \cdot B$ (that is to say, $P = i\omega H^* \cdot B$, if one sets $H = \nu_{eff} B$), with an effective reluctivity ν_{eff} that is not only anisotropic (i.e., a tensor), and complex-valued (because of inevitable Joule losses), but can exhibit a *negative* real part along some directions (here, vertically), in some narrow window $[\omega_1, \omega_2]$ of angular frequencies. This is due to an "internal resonance" when $LC \sim \omega^{-2}$, where L is the ring inductance and C the slit capacitance. All it takes is a near-perfect conductor (or dielectric), in order to have $e \sim 0$ in the ring. Then, by Faraday's law, a displacement current must cross the slit, hence a large electric field there, whose average energy can offset the cell's magnetic energy if ω is slightly above the resonant frequency.

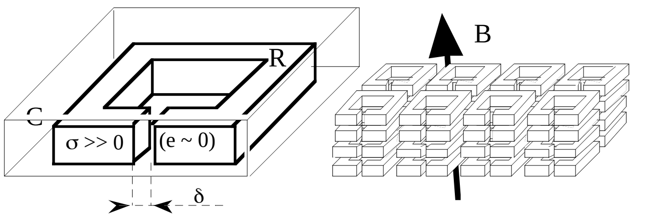

Fig. 1. A "split ring" R in its symmetry cell C (left), and a stack of such cells, making a metamaterial (right). An oversimplistic design, as such materials go, but enough to demonstrate our frequency-dependent homogenization technique. Later, the thin slit of width δ is replaced by a surface Σ, and the air region $C - (R \cup \Sigma)$ is denoted A.

Our objective is to define a so-called "cell problem", a boundary value problem (BVP) derived from the Maxwell equations, but set on C only instead of on the entire space, which once solved would provide ν_{eff}, or its inverse μ_{eff}, and also (though we shall not go into this in detail) the analogous ε_{eff}. From this point on, one would be free to solve the macroscopic problem at hand (with a specific shape for the metamaterial-filled region(s), specific source currents, etc.) by using the effective coefficients only, ignoring the microstructure that has already been dealt with when solving the cell problem(s). This will allow a cheaper numerical simulation (by allowing the use of much coarser finite elements nets, for instance), and perhaps more importantly, serve in *designing* metamaterials by repetitively solving the cell problem inside some iterative procedure which makes the cell's structure evolve towards some optimal one.

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This procedure—solve a micro-scale cell problem, or several, then address a macro-scale problem where all materials are homogeneous, with effective coefficients as found—is called *homogenization* [1]. As an approximation technique (which it is: one replaces the original problem by a different one, deemed close to it), it requires a justification, that only some appropriate convergence theorem can provide. This is our main concern in the present work. But let it be clear that *which* theorem to prove (a matter of asking the right question, that is, an act in *modelling*), is more important here than *proving* it, a technical matter, which we shall be able to keep much simpler than in other approaches to homogenization [3] by assuming *linearity* of the constitutive laws. This allows one to use standard harmonic analysis techniques such as (spatial) Fourier transform and Floquet–Bloch decomposition [4], whose relatively involved interrelations provide the gist of the proof technique.

About notation: We shall need to distinguish *vectors*, elements of the 3D vector space V_3, and *points*, elements of the associated affine space, denoted A_3. If x is a point and v a vector, $x + v$ denotes a point, the *translate* of x by v. V_3 is an additive group, which thus *acts* on A_3 "by translations". (Italics, when not used only for emphasis, signal that some implicit definition is being given, or suggested, as here.)

Section II recalls the Bloch–Floquet transform. Its analogies and differences with Fourier are addressed in Section III, which suggests that Fourier analysis is a limit case of Floquet–Bloch, as obtained when the size of the symmetry cell tends to 0. Section IV uses this to establish the main convergence result about homogenization in magnetostatics. In Section V, the same kind of result is obtained for the full Maxwell system (in steady-state regime at frequency ω), but is found, though correct, inadequate as regards modelling. Section VI solves the conundrum this raises in the case of the split-ring metamaterial by introducing a second parameter, besides the cell's size, in the analysis, namely the slit's width. When both parameters go to zero simultaneously in a definite relationship (the slit's width should scale like the *cube* of the cell's size), the convergence theorem points to the expected cell problem, an exotic variant of magnetostatics on the cell, able to take into account capacitive effects. A particular case of this problem proves amenable to analytical treatment, which confirms the ability of homogenization theory to predict a frequency window of negative effective permeability.

II. Floquet–Bloch Analysis

Homogenization is the exploitation of symmetry, namely translational symmetry of the crystal-like structure, taking account of the smallness of the symmetry cell. These are two distinct issues. Let's address the first one to begin with.

Suppose one wants to solve $-\mathrm{div}(\mu\ \nabla\varphi) = q$ over all space (a well-posed problem if $q \in L^2(A_3)$ has bounded support), where $\mu(x + \tau) = \mu(x)$ for all x and all translation vectors τ of the form $\sum z^i\, v_i$, where the z^i are relative integers. This way, the distribution of μ in all space is determined by its pattern over a *symmetry cell* C, defined up to translation, comprised of all points of the form $x = c + \sum_i \lambda^i\, v_i$ with $|\lambda^i| \le 1/2$, where v_1, v_2, v_3 are three fixed independent vectors. C is thus centered at some point c that we may use later as origin. The translates of C by all vectors τ of the above form pave space, with overlap at their boundaries. The set T of such translation vectors, which form a group of mappings from A_3 to itself (a subgroup of V_3, isomorphic to Z^3), is called the *Bravais lattice.*

The *Floquet–Bloch decomposition* of a function φ consists in writing it as a sum

$$\varphi(x) = (2\pi)^{-3} \int_B d\kappa\, \exp(i\, \kappa \cdot x)\, \hat{\varphi}_\kappa(x). \qquad (1)$$

over some subset B of V_3 that we will describe in a moment. Vectors κ are called *wavevectors*. (Note the dual use of symbol x here: in $\kappa \cdot x$, it denotes the vector from some point designated as origin, for instance, the cell's center c, to point x.) Functions $\hat{\varphi}_\kappa(x)$ live on C and are *C-periodic,* i.e., take the same value at point pairs x and $x + \tau$ of ∂C linked by one of the translations in T. These so-called *Bloch modes* are obtained by the summation

$$\hat{\varphi}_\kappa(x) = \mathrm{vol}(C) \sum_{\tau \in T} \exp(-i\kappa \cdot (x + \tau))\, \varphi(x + \tau). \qquad (2)$$

The analogy with the Fourier transform formulas, direct,

$$\hat{\varphi}(\kappa) = \int_{V_3} d\tau\, \exp(-\, i\, \kappa \cdot \tau)\ \varphi(\tau), \qquad (3)$$

and inverse,

$$\varphi(x) = (2\pi)^{-3} \int_{V_3} d\kappa\, \exp(i\, \kappa \cdot x)\, \hat{\varphi}(\kappa), \qquad (4)$$

may help understand why (1)(2) holds, but some group theory would be necessary to unfold this theory. Let us just mention that functions such as $\chi_\kappa(\tau) = \exp(i\kappa \cdot \tau)$, defined on the group T of translations (which is the full vector space V_3 in the case of Fourier) are called *characters,* their defining property being $\chi_\kappa(\tau + \theta) = \chi_\kappa(\tau)\, \chi_\kappa(\theta)$, and that the integration in (1) is over the set B of such characters. To identify this set, let us call w_1, w_2, w_3 the *dual* vectors to the v_is, defined by $w_i \cdot v_i = 2\pi$ and $w_j \cdot v_i = 0$ if $i \ne j$, and B (akin to the "Brillouin zone" of crystallography) the cell built on the w_is the same way C was built on the v_is. Translates of B by vectors $\theta = \sum_i z^i\, w_i$, with integer weights z^i, pave space. Since $\exp(i\kappa \cdot \tau) = \exp(i\, (\kappa + \theta) \cdot \tau)$, one gets all the characters by letting κ span B only.

Bloch analysis consists in throwing (1) into the equation to be solved, which results (thanks to the invariance of μ under translations of T, in our case) in a family of subproblems of the same kind, one for each κ. This is very useful when there is a *single* wavevector κ in sight, for plane wave scattering problems, for instance. But otherwise, prospects of saving on computation this way look grim, a priori: What is the point of replacing a single partial differential equation (even though its domain is the whole space), by an infinity of BVP on C which—depending on κ by their boundary conditions, as we shall see they do—are all different? We note that the treatment by the Fourier transform, when applicable, that is, when μ is invariant by *all* translations, does not suffer from the same shortcomings, because it replaces the original equation by an infinity of *algebraic* ones (namely, $\kappa \cdot (\mu\ \hat{\varphi}(\kappa)\ \kappa) = \hat{j}(\kappa)$, for all $\kappa \in V_3$, where the Fourier component $\hat{j}(\kappa)$ and $\hat{\varphi}(\kappa)$ of j and φ are complex *numbers*, rather than *functions*). These equations can be solved in one stroke, hence the solution φ by inverse Fourier transform. What homogenization does, as we shall see, is reduce the complexity of the Floquet–Bloch treatment by solving (approximately, of course) all subproblems in one stroke too.

III. Bloch and Fourier

The right-hand side (r.h.s.) of (1) looks like an approximate quadrature formula for the r.h.s. of (3), and (4) looks like what (2) would be, should the cell C reduce to a point and B grow up to fill space entirely. To make rigorous sense of these observations, let us consider a family of Bravais lattices T_α, indexed by a real $\alpha > 0$, generated by the translation vectors αv_1, αv_2, αv_3, and look at what happens when $\alpha \to 0$. The homothetic image C_α of C, with center c and ratio α, makes a symmetry cell for T_α, and the corresponding set of characters is $B_\alpha = \alpha^{-1} B$. We denote by $\langle\hat{\varphi}_\kappa\rangle_\alpha$ the average of $\hat{\varphi}_\kappa$ over C_α. One easily proves that

Lemma 1. *Given* φ *in* $L^2(A_3)$, $\langle\hat{\varphi}_\kappa\rangle_\alpha \to \hat{\varphi}(\kappa)$ *as* $\alpha \to 0$.

There is a kind of reciprocal result, the not too difficult, but technical proof of which we omit:

Lemma 2: *Given* φ *and a bounded family* $\{\varphi^\alpha : \alpha > 0\}$ *of functions in* $L^2(A_3)$, *assume that* $\langle\hat{\varphi}^\alpha{}_\kappa\rangle_\alpha \to \hat{\varphi}(\kappa)$ *for all* κ. *Then* φ^α weakly *converges to* φ *when* $\alpha \to 0$.

Weak convergence means that weighted averages of φ^α converge towards those of φ. Remark than no more is expected from homogenization than good approximation of spatial *averages* of fields, so this looks fine.

IV. Static Homogenization: Magnetostatics

Rather than committing ourselves to the use of a scalar potential φ as we did thus far, it will be more convenient to deal with the problem in the following symmetric form: Given μ, invariant by all translations of lattice T, and given the compactly supported source current j, find fields h and b such that

$$\text{div } b = 0, \quad b = \mu h, \quad \text{rot } h = j. \tag{5}$$

This is "problem P". We shall search for the Floquet–Bloch decompositions of b and h, that is, for functions $\hat{h}_\kappa$, $\hat{b}_\kappa$, living on the symmetry cell C, such that $h(x) = (2\pi)^{-3}\int_{V_3} d\kappa \exp(i\,\kappa \cdot x)\, \hat{h}_\kappa(x)$, etc. Since $\text{rot}[\exp(i\kappa \cdot x)\, \hat{h}_\kappa(x)] = \exp(i\,\kappa \cdot x)[\text{rot } + i\,\kappa\, \times]\, \hat{h}_\kappa(x)$, with obvious notation, and a similar thing about div, one has "problem P_κ" to solve,

$$(\text{div} + i\,\kappa\, \cdot)\, \hat{b}_\kappa = 0,\ \hat{b}_\kappa = \mu\, \hat{h}_\kappa,\ (\text{rot} + i\,\kappa\, \times)\, \hat{h}_\kappa = \hat{j}_\kappa, \tag{6}$$

for each $\kappa \in B$, on C, with periodic boundary conditions, where the $\hat{j}_\kappa$'s are the Bloch modes of j. Notice how the multiplication by $\exp(-i\,\kappa \cdot x)$, while uniformizing the boundary conditions for this κ-indexed infinity of cell problems, moved the dependence on κ into the differential operators. Thus (6) is by no means simpler than (5), quite the contrary.

But now comes the decisive move, by which we exploit the smallness of the cell C. Let us embed problem P into a family P^α of similar ones, but with C_α-periodicity of μ instead of C-periodicity, so that (5) is problem P^1, one among the family. The corresponding cell problems P^α_κ are the same as in (6), apart from the α indexes, and the C_α- instead of C-periodicity. As this last feature prohibits a comparison between the P^α_κ's for a fixed κ and different α's, we "pull back" problems P^α_κ onto C by the transform $x \to c + (x - c)/\alpha$, which blows up C_α to C. Denoting by b^α_κ, h^α_κ, etc., hats dropped, the pullbacks (e.g.: $b^\alpha_\kappa(y) = \hat{b}^\alpha_\kappa(c + \alpha(y - c))$, thus living on C, and C-periodic), one sees that

$$(\text{div} + i\,\alpha\kappa\, \cdot)\, b^\alpha_\kappa = 0, \qquad (\text{rot} + i\,\alpha\kappa\, \times)\, h^\alpha_\kappa = \alpha\, j^\alpha_\kappa,$$
$$b^\alpha_\kappa = \mu\, h^\alpha_\kappa, \tag{7}$$

with now the same μ for all (it's the original μ, C-periodic), and the prospect to see all these κ-subproblems become "the same" when $\alpha \to 0$.

Proposition 1. *When* $\alpha \to 0$, *the solution* $\{b^\alpha_\kappa, h^\alpha_\kappa\}$ *of* (7) *converges* (in the strong sense of $L^2(C)$) *towards the solution* $\{b_\kappa, h_\kappa\}$ *of*

$$\text{div } b = 0, \qquad b = \mu h, \qquad \text{rot } h = 0 \tag{8}$$

$$i\,\kappa \cdot \langle b\rangle = 0, \qquad i\,\kappa \times \langle h\rangle = \hat{j}(\kappa). \tag{9}$$

where $\langle\ \rangle$ *denotes averaging over* C.

Proof (sketched). Terms in α vanish, hence (8). Since b^α_κ and h^α_κ are C-periodic, the integrals $\int_C \text{div } b^\alpha_\kappa$ and $\int_C \text{rot } h^\alpha_\kappa$ are null. Integrate the first line of (7) over C, and use Lemma 1 to find the r.h.s. $\hat{j}(\kappa)$ in (9). ◊

Now look at (8)(9). This problem still depends on κ, but it splits into two parts. One is "find the relation $B = \mu_{eff} H$ that must exist between vectors B and H for

$$\text{div } b = 0,\ \langle b\rangle = B,\ b = \mu h,\quad \text{rot } h = 0,\ \langle h\rangle = H \tag{10}$$

to have a solution", and this is the expected *cell-problem*, from which κ has disappeared. The second part consists in finding, for each κ, vectors $\langle b\rangle$ and $\langle h\rangle$ (complex valued) such that

$$i\,\kappa \cdot \langle b\rangle = 0, \quad \langle b\rangle = \mu_{eff}\, \langle h\rangle, \quad i\,\kappa \times \langle h\rangle = \hat{j}(\kappa), \tag{11}$$

and here we recognize the κ-indexed algebraic problems associated with the Fourier method of solving

$$\text{div } b = 0, \qquad b = \mu_{eff}\, h, \qquad \text{rot } h = j. \tag{12}$$

Thanks to Lemma 2 (and the fact that the averages of a field over C_α and of its pullback over C are equal), we may conclude as follows:

Theorem 1. *When* $\alpha \to 0$, *the solution* $\{b^\alpha, h^\alpha\}$ *of problem* P^α *weakly converges towards the solution* $\{b, h\}$ *of* (12), *where the homogenized permeability* μ_{eff} *is the one obtained by solving the cell problem* (10).

Does this justify replacing problem P^1, the original one, by P^0, if we so label problem (12)? As with all perturbative techniques, this depends on the magnitude of the terms thus neglected in the Taylor expansion in α of $\{b^\alpha, h^\alpha\}$ near $\alpha = 0$. It can be shown (but this is another issue than what Thm 1 addresses) that this is so if the spectral content of the spatial Fourier transform of j is poor in "small wavelengths", where small refers to the size L of cell C, i.e., if $\hat{j}(\kappa)$ is small enough to be neglected when $L|\kappa| << 1$ does not hold. To appreciate whether this is so pertains to the "modelling" part of the procedure.

V. Homogenizing The Full Maxwell System

Let us now try the same approach on the Maxwell equations

$$-i\omega\, \varepsilon e + \text{rot } h = j, \quad i\omega\, \mu h + \text{rot } e = 0,$$

where ε integrates the conductivity σ by the usual trick of setting $\varepsilon = \varepsilon_0 - i\sigma/\omega$. Our work is cut out for us: Proceeding exactly as above, we find this instead of (7):

$$-i\omega\, \alpha d^\alpha_\kappa + (\text{rot} + i\,\alpha\kappa\, \times) h^\alpha_\kappa = \alpha\, j^\alpha_\kappa, \quad d^\alpha_\kappa = \varepsilon\, e^\alpha_\kappa,$$
$$i\omega\, \alpha b^\alpha_\kappa + (\text{rot} + i\,\alpha\kappa\, \times) e^\alpha_\kappa = 0, \quad b^\alpha_\kappa = \mu\, h^\alpha_\kappa,$$

and the $\alpha = 0$ limit is characterized by

$$\text{div } b = 0, \qquad b = \mu h, \qquad \text{rot } h = 0,$$
$$\text{div } d = 0, \qquad d = \varepsilon\, \varepsilon, \qquad \text{rot } e = 0,$$
$$-i\omega\, \langle d\rangle + i\,\kappa \times \langle h\rangle = \hat{j}(\kappa), \quad i\omega\, \langle b\rangle + i\,\kappa \times \langle e\rangle = 0. \tag{13}$$

The weak limit, the same theorem tells us, will satisfy (13) with constitutive relations $\langle b\rangle = \mu_{eff}\, \langle h\rangle$ and $\langle d\rangle = \varepsilon_{eff}\, \langle e\rangle$, which are the image in Fourier space of the Maxwell equations for a material with effective coefficients μ_{eff} and ε_{eff}, as given by solving the cell problem (10) and its electrostatic counterpart.

But this is a big disappointment! We *don't* expect the effective coefficients for a metamaterial to be independent of frequency, as this result would tend to suggest, especially not in the case of Fig. 1. What went wrong? Not the logic underlying the theorem, but the modelling: The choice of problems P^α in which to embed P^1 cannot be the same as in statics, because it must take into account the existence, besides the cell's size, of another small parameter, the width of the slit (to say nothing of the penetration depth, which we take null from the onset by assuming $e = 0$ in the bulk of the ring).

VI. Homogenizing The Split-Ring Metamaterial

Indeed, since the resonance condition $LC \sim \omega^{-2}$ is essential, it should be an invariant feature of all problems P^α in order to be preserved at the limit. When $\alpha \to 0$, both L and C scale like α if the cell is shrunk by homothetic contraction, hence the resonance is lost, so one should imagine a family of problems P^α that make the slit capacitance behave in $1/\alpha$. This is easy: Take the slit width in P^α equal to $\alpha^3\delta$. Let us now look for the $\alpha = 0$ limit under these conditions.

To keep things simple, we start from a modelling of problem P^1 that already acknowledges the high conductivity of the ring (this is done by assuming $e = 0$ in R) and the smallness of δ (this is done by modelling the slit by a surface Σ that bears a capacitive layer). Cell C is made of the ring R, the slit Σ, and the "air part" A around. With hopefully little risk of confusion, we also denote by R, Σ, A the set-unions of all translates of these cell parts in the given lattice. The same conventions apply, with index α appended, for problems P^α. Now, the version of problem P^α we propose ourselves is, in weak form, *find* h^α *such that*

$$\int_{A_\alpha} i\omega\mu\, h^\alpha \cdot h' + \int_{A_\alpha} (i\omega\varepsilon)^{-1} (\mathrm{rot}\, h^\alpha - j) \cdot \mathrm{rot}\, h' \qquad (14)$$
$$+ \int_{\Sigma_\alpha} (i\omega\varepsilon)^{-1} \alpha^3 \delta\, (n \cdot \mathrm{rot}\, h^\alpha)\, (n \cdot \mathrm{rot}\, h') = 0$$

for all test fields h', where n denotes the unit normal on Σ_α. Notice how (in the case $\alpha = 1$) the third term accounts for the capacitive effect across Σ.

We proceed as above, by throwing the Bloch decomposition of h into (14), hence the κ-indexed cell problems, then scaling to pull them back to the reference cell C. This results in *find* h^α_κ (a Bloch mode, now), C-*periodic, such that*

$$\alpha^3 \int_A i\omega\mu\, h^\alpha \cdot h'$$
$$+ \alpha \int_A (i\omega\varepsilon)^{-1} ((\mathrm{rot} + i\alpha\kappa\,\times)\, h^\alpha - \alpha j) \cdot (\mathrm{rot} + i\alpha\kappa\,\times)\, h'$$
$$+ \alpha^3 \int_\Sigma (i\omega\varepsilon)^{-1} \delta\, [n \cdot (\mathrm{rot} + i\alpha\kappa\,\times)\, h^\alpha][n \cdot (\mathrm{rot} + i\alpha\kappa\,\times)\, h'] = 0$$

for all test fields h', where one clearly sees how the α^3 term in front of δ maintains the balance between inductive and capacitive reactive powers. The middle line acts like a penalty term, making rot h equal to 0 in the $\alpha = 0$ limit.

Let us now, leaving intermediate details aside, describe this limit. Since $\mathrm{rot}\, h = 0$ in A, we write it $h = \mathrm{grad}\, \varphi$, with φ possibly multivalued, since a current I will flow in the ring across Σ. Let us therefore introduce a cutting surface S with respect to the loop that R makes, across which φ may have a jump $[\varphi]$, equal to I. Jumps can also exist on ∂C, because although h is C-periodic, φ need not be: What one must have is just $\varphi(x + v_i) - \varphi(x)$ equal to some constant c_i for pairs of boundary points x and $x + v_i$. Call Φ the space of admissible potentials under these conditions. The weak form of the cell problem then turns out to be, *find* φ *in* Φ *such that, for all test functions* φ' *in* Φ,

$$\int_A i\omega\mu\, \nabla\varphi \cdot \nabla\varphi' + (i\omega C)^{-1} [\varphi][\varphi'] = \int_A i\omega\, B \cdot \nabla\varphi' \quad (15)$$

where C, the capacitance of the slit, is taken equal to $\int_\Sigma \varepsilon_0/\delta$, and B, a given vector parameter, can be recognized as the average induction. To find the effective reluctivity, as suggested in the Introduction, we then set

$$\nu_{eff}\, B \cdot B = \int_A \mu\, |\nabla\varphi|^2 - 1/(C\omega^2)\, |[\varphi]|^2.$$

Note how ν_{eff} can change sign. It is *real*, here, owing to our perfect conductor assumption, which excludes losses, but it would be easy to correct (15) to account for these: Just add a "boundary impedance" term, in $(1 + i)(\sigma d)^{-1}\, h \cdot h'$, where d is the skin depth, on ∂R.

This is a mildly exotic variant of the magnetostatics problem, easy to solve by using nodal elements for φ in A (and node-doubling on the cut S). If one focuses interest on the vertical part of ν_{eff} (B then being vertical), and if the height of the ring almost matches that of the cell, the two-dimensional situation that results is amenable to analytical calculation (Fig. 3). One finds

$$\nu_{eff}(\omega) = \nu_0\, (1 - \omega^2 \mu_0\, aC)/[a + a'(1 - \omega^2 \mu_0\, aC)], \qquad (16)$$

where a and a' are the *relative* (to the cross section) areas of the two parts of the air region and C the slit's capacitance. Hence a window $\omega_1 = \nu_0/aC$ to $\omega_2 = \nu_0\,(a + a')/aa'C$ of negative effective reluctivity. For σ finite but very large, correcting for skin effect as explained above, one can expect the result plotted on Fig. 3, right.

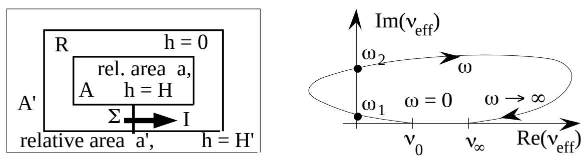

Fig. 3. Left: Horizontal cross section of C. The ring current I is $H - H'$ (by Ampère), one has $I = i\omega CV$, where V is the voltage drop across the gap, and V relates to H by Faraday. Hence (16) follows. Right: Plot of ν_{eff} in the complex plane, parameterized by the angular frequency, if Joule losses are accounted for.

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