ON THE HOMOGENIZATION OF MAXWELL EQUATIONS

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We discuss the following general problem: given a composite, made from at least two different materials with each its own (scalar, and possibly complex-valued) ε and μ, distributed in space in a regular, crystal-like, pattern, find the *equivalent* permittivity and permeability (they will, in general, be tensors). This is *homogenization*. A computation in which such repetitive composites are present can then be done by simply replacing these with their equivalent, homogenized, materials.

Homogenization, however, is a *frequency-dependent* process, and the point of the present article is to stress this, which seems to have been overlooked by applied mathematicians. Intuitively, it makes sense: a finely-meshed conductive grid may, for instance, be impervious to micro-waves, and effectively shield against them, whereas one can see through it. So its "effective penetration depth", that is, the penetration depth of its equivalent material, depends on frequency. Most theoretical studies on homogenization theory, however, seem to ignore this dependence. We wish to present here an approach in which full attention is given to this problem.

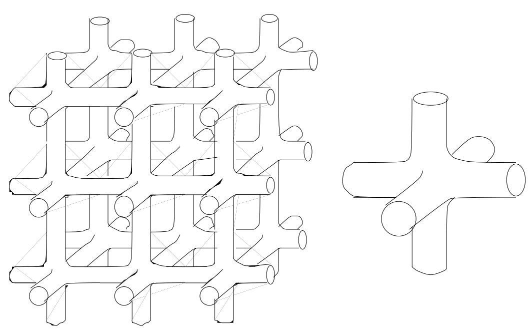


Fig. 1. Repetitive, conductive grid. The equivalent homogenized medium is opaque to waves in the GHz range: it behaves as if there was an equivalent skin-depth, small, but quite large with respect to the conventional skin-depth in the metal. On the right, the basic cell.

As a model problem, consider the regular grid of Fig. 1. Assume a wavelength of about 12 cm (this is the "large scale"), and a size of the basic brick in Fig. 1, left (referred to, from now on, as "the cell"), of a few millimeters (this is the "cell scale"). If one is interested in the absorptive properties of such a periodic medium [2, 3], the relevant question is therefore to homogenize Maxwell equations,

$$- i\omega d + \mathrm{rot}\, h = j^s, \quad i\omega b + \mathrm{rot}\, e = 0, \tag{1}$$

$$d = \varepsilon\, e \equiv (\varepsilon_0 - \sigma/i\omega)\, e, \quad b = \mu h, \tag{2}$$

where σ is the conductivity of the metal, and j^s some "source"-current (not necessarily divergence-free). As a result, we expect the large-scale behavior of the field to be describable by the following equations:

$$-\,i\omega D + \mathrm{rot}\ H = j^s, \quad i\omega B + \mathrm{rot}\ E = 0 \qquad (3), \qquad \{D, B\} = M(\omega)\ \{E, H\}. \qquad (4)$$

Here, the so-called "large-scale fields" D, B, E, H are the spatial averages, across several cells, of the actual fields d, b, e, h. They thus result from "filtering out" the cell-scale variations of d, b, e, h, that is, their high-frequency (spacewise) components. As for the 6×6 complex matrix M, it describes the homogenized constitutive laws. One might expect M to be block-diagonal, with "equivalent", tensorial, $\underline{\varepsilon}$ and $\underline{\mu}$ as its diagonal blocks, but although this is certainly what happens at zero frequency, the general case is different: one may not a priori rule out the case when the other two blocks would differ from 0.

Let us assume Maxwell equations (1)(2) are to be solved in some spatial domain D, bounded by a surface S, and filled with a repetitive pattern of conductor and air, as suggeted by Fig. 1. For definiteness, suppose $j^s = 0$, the source of the field being in the boundary conditions on S. (What these are is irrelevant to homogenization, but of course, once $M(\omega)$ has been obtained, these same boundary conditions will apply to (3)(4).) To specify the spatial periodicity of σ, let us introduce three fixed, independent, spatial vectors $\partial_1, \partial_2, \partial_3$, and a typical length L, in such a way that

$$\sigma(x \pm L\ \partial_i) = \sigma(x),\ \forall\ x\ \text{in}\ D_0,\ i = 1, 2, 3, \qquad (5)$$

where D_0 is a part of D that contains all points farther from S than some specified distance d, larger than L. (Near the boundary, the behavior of σ has no special symmetry, hence the restriction to D_0.) This implies a similar "quasi-periodicity" condition about ε, and we may as well, for the sake of generality, assume the same about μ. We call "cell at x", denoted by C_x, the set of all points of the form $x + \sum_{i = 1, 2, 3} L\ \theta_i\ \partial_i$ for $-{}^1\!/_2 \le \theta_i \le {}^1\!/_2$. We may always select coordinates in such a way that the origin is in D_0. We then call C the cell at 0. We say a field u is C-*periodic* iff it satisfies the following condition (stronger than (5))

$$u(x + L\ \partial_i) = u(x),\ \forall\ x\ \text{in}\ E_3,\ i = 1, 2, 3, \qquad (6)$$

where E_3 denotes the whole Euclidean three-dimensional space. Although the problem is to be solved over D, we make once and for all the convention that all fields are over all E_3, by assigning to them the value 0 outside D.

Let χ be a "mollifier", that is, a smooth non-negative function, invariant by rotations around 0, satisfying $\int \chi = 1$, with a support $\mathrm{supp}(\chi)$ centered at 0 and of small diameter, though large with respect to the cell-size. Let us define $E = \chi * e$, where $*$ denotes the convolution product, and similarly, H, D and B. Since B and E have small variations over the cell, we may substitute truncated Taylor expansions for them: $B(x) = \mathrm{B} + (...)$, etc., where B is the value of B at the origin. Since, by linearity, $i\omega B + \mathrm{rot}\ E = 0$ must hold, the shorter limited expansions that match are $E(x) = \mathrm{E} - {}^1\!/_2\ i\omega\mathrm{B} \times x + (...)$, $B(x) = \mathrm{B} + (...)$, and similarly, $H(x) = \mathrm{H} + {}^1\!/_2\ i\omega\mathrm{D} \times x$ and $D(x) = \mathrm{D} + (...)$, where the dots denote the remainders.

Now, *if* D was the whole space, one would have, by a straightforward Fourier decomposition, $e = E + e_C$ and $h = H + h_C$, where e_C and h_C are C-periodic, and satisfy

$$\int_C e_C = 0, \qquad \int_C h_C = 0. \qquad (7)$$

For a bounded D, of course, such a simple decomposition does not hold, and we only have $e = E + e_C + e_R$ and $h = H + h_C + h_R$, where e_C and h_C are C-periodic fields that satisfy (7), e_R and h_R being "residual" fields of small (vanishing with L) magnitude inside D_0. But this remark suggests a powerful heuristic: we'll *neglect* these residuals, as well as the higher order terms in the previous expansions. Let us thus replace e and h in (1) with $\mathrm{E} - {}^1/_2\, i\omega \mathrm{B} \times x + e_C$ and $\mathrm{H} + {}^1/_2\, i\omega \mathrm{D} \times x + h_C$, where e_C and h_C are C-periodic. This yields

$$-\, i\omega\varepsilon\, e_C(x) + \mathrm{rot}\, h_C(x) = -i\omega(\mathrm{D} - \varepsilon \mathrm{E}) + \frac{\varepsilon\omega^2}{2}\, \mathrm{B} \times x, \tag{8}$$

$$i\omega\mu\, h_C(x) + \mathrm{rot}\, e_C(x) = i\omega(\mathrm{B} - \mu \mathrm{H}) + \frac{\mu\omega^2}{2}\, \mathrm{D} \times x, \tag{9}$$

a boundary-value problem over the cell C, set in terms of e_C and h_C, subject to (7) and to periodicity conditions. There are four parameters in this problem, namely the three-dimensional complex vectors D, B, E and H, and not any combination of such parameters will give a solution: as will be clear later, they should satisfy a compatibility condition of the form $\{\mathrm{D}, \mathrm{B}\} = M(\omega)\, \{\mathrm{E}, \mathrm{H}\}$, which is precisely the homogenized constitutive law we are looking for, since D, B, E and H are the averages, over the cell, of the large-scale fields D, B, E, H.

Before going on, let us put this statement to a simple test: in the case of an homogeneous material (ε and μ constant over C), one should have $\mathrm{D} = \varepsilon \mathrm{E}$ and $\mathrm{B} = \mu \mathrm{H}$ as compatibility conditions. Indeed, this is what comes from averaging (8) and (9) over C, for the average of terms like $x \to \mathrm{B} \times x$ is 0, since the cell is centered at the origin.

Note that $e_C + \mathrm{E}$ and $h_C + \mathrm{H}$ are C-periodic fields also, and that their averages over the cell are E and H. This remark allows us to present the previous problem in a slightly simplified form, *find* C-*periodic fields* e_C *and* h_C (no more subject to (7)) *such that*

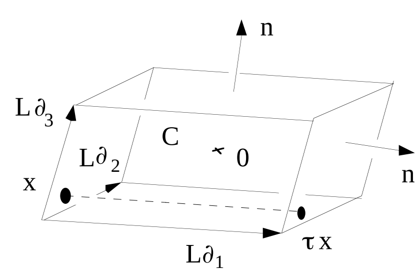
$$-\, i\omega\varepsilon\, e_C(x) + \mathrm{rot}\, h_C(x) = -i\omega \mathrm{D} + \frac{\varepsilon\omega^2}{2}\, \mathrm{B} \times x, \tag{10}$$

$$i\omega\mu\, h_C(x) + \mathrm{rot}\, e_C(x) = i\omega \mathrm{B} + \frac{\mu\omega^2}{2}\, \mathrm{D} \times x, \tag{11}$$

then set $\mathrm{E} = (\int_C e_C)/\mathrm{vol}(C)$ *and* $\mathrm{H} = (\int_C h_C)/\mathrm{vol}(C)$, hence $M(\omega)$, or rather, its inverse. This we shall call the "cell-problem". No more compatibility condition now: D and B clearly appear as source-terms in the cell-problem, which yields the corresponding E and H plus, if needed, the *local* fields at the cell-scale, $e_C - {}^1/_2\, i\omega \mathrm{B} \times x$ and $h_C + {}^1/_2\, i\omega \mathrm{D} \times x$.

In practice, the cell-problem (10)(11) will be solved via finite elements, so let us give a weak formulation for it. Using the standard notation $\mathbb{L}^2(C)$ for square-integrable complex-valued vector fields over C, let us set $\mathbb{L}^2_{rot}(C) = \{u \in \mathbb{L}^2(C) : \mathrm{rot}\, u \in \mathbb{L}^2(C)\}$, and intro- duce the following, ad-hoc notion (Fig. 2): the *warp* of $u \in \mathbb{L}^2_{rot}(C)$ across the cell, denoted by $[n \times u]$, is the tangential field

$$x \to n(x) \times u(x) + n(\tau x) \times u(\tau x)$$



defined on the boundary ∂C of C, where (cf. inset) τx denotes the image of x by one of the translations $\pm L\partial_i$, and n the outer unit normal. (There is a unique such translation for all points in ∂C, except at corners, so the warp is properly defined, almost-everywhere on ∂C.) We may then define the functional spaces $\mathbb{E}_C \equiv \mathbb{H}_C = \{u \in \mathbb{L}^2_{rot}(C) : [n \times u] = 0\}$.

Fields in $\mathbb{E}_C$ and $\mathbb{H}_C$ ("warp-free" fields) are obviously in one-to-one correspondence with the above mentioned C-periodic fields. Now take the scalar product of both sides of both equations (10) and (11) by warp-free test-fields e' and h', and integrate over C. This results in

$$-\,i\omega\int_C \varepsilon\, e_C \,.\, e' + \int_C h_C \,.\, \mathrm{rot}\, e' = -i\omega\int_C \mathrm{D} \,.\, e' + \int_C \frac{\varepsilon\omega^2}{2}\, \mathrm{B}\times x \,.\, e'(x)\, dx,$$

$$i\omega\int_C \mu\, h_C \,.\, h' + \int_C e_C \,.\, \mathrm{rot}\, h' = i\omega\int_C \mathrm{B} \,.\, h' + \int_C \frac{\mu\omega^2}{2}\, \mathrm{D}\times x \,.\, h'(x)\, dx,$$

where dx denotes the volume element. We find two complementary (and equivalent) weak formulations by eliminating either e_C or h_C, hence the two problems, *find* e_C *in* $\mathbb{E}_C$ *such that, for all* e' *in* $\mathbb{E}_C$,

$$i\omega\int_C \varepsilon\, e_C \,.\, e' + \int_C (i\omega\mu)^{-1}\, \mathrm{rot}\, e_C \,.\, \mathrm{rot}\, e' = \int_C \mu^{-1}\, \mathrm{B} \,.\, \mathrm{rot}\, e' - \int_C i\omega/2\ \mathrm{D}\times x \,.\, \mathrm{rot}\, e'(x)\, dx$$
$$+\, i\omega\int_C \mathrm{D} \,.\, e' - \int_C \frac{\varepsilon\omega^2}{2}\, \mathrm{B}\times x \,.\, e'(x)\, dx, \qquad (12)$$

and the similar one in h, *find* h_C *in* $\mathbb{H}_C$ *such that, for all* h' *in* $\mathbb{H}_C$,

$$i\omega\int_C \mu\, h_C \,.\, h' + \int_C (i\omega\varepsilon)^{-1}\, \mathrm{rot}\, h_C \,.\, \mathrm{rot}\, h' = -\int_C \varepsilon^{-1}\, \mathrm{D} \,.\, \mathrm{rot}\, h' - \int_C i\omega/2\ \mathrm{B}\times x \,.\, \mathrm{rot}\, h'(x)\, dx$$
$$+\, i\omega\int_C \mathrm{B} \,.\, h' + \int_C \frac{\mu\omega^2}{2}\, \mathrm{D}\times x \,.\, h'(x)\, dx. \qquad (13)$$

Both problems have a unique solution, unless ω happens to be a resonant frequency of C, but that is not possible if $\sigma > 0$ somewhere [1]. Their solution via edge-elements is straightforward (the periodicity condition is especially easy to implement: just assign the same degree of freedom to a boundary edge e and to its translate by τ). For the computation of E and H, look at (10) and (11): by averaging the equality $e_C(x) = (i\omega\varepsilon)^{-1}\, \mathrm{rot}\, h_C(x) + \varepsilon^{-1}\, \mathrm{D} + i\omega/2\ \mathrm{B}\times x$ and the similar one about h_C, we find

$$\mathrm{E} = [\textstyle\int_C (i\omega\varepsilon)^{-1}\, \mathrm{rot}\, h_C + \int_C \varepsilon^{-1}\, \mathrm{D}]/\mathrm{vol}(C), \quad \mathrm{H} = [-\int_C (i\omega\mu)^{-1}\, \mathrm{rot}\, e_C + \int_C \mu^{-1}\, \mathrm{B}]/\mathrm{vol}(C). \qquad (14)$$

When one does all this, it may come as a surprise to see "cross-dependencies": B, e.g., depends not only on H but on E as well, and D depends both on E and H. This kind of behavior, which is characteristic of the so-called *chiral* materials (because they rotate the polarization plane of waves), may result from homogenization, even if the constituents are not chiral themselves. What is required for this to happen is some *geometrical* chirality at the cell level, that is, the cell should fail to be congruent with its image by central symmetry. So homogenized media are also, as a rule, chiral media.

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